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YADE 1D vertical VANS fluid resolution: Theoretical basis

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ABSTRACT

In the present note, we discuss the theoretical basis of models coupling Discrete Element Method (DEM) with Volume-Averaged Navier-Stokes (VANS) equations. Starting from the classical equations derived in Anderson & Jackson (1), the different closures for the total fluid stress tensor and the fluid-particle momentum transfer are discussed from a physical meaning point of view. This allows to avoid to take into account two times the same effect and clarify the meaning of the different terms.

Keywords: Yade; Volume-averaged Navier-Stokes equations; theoretical basis; fluid-DEM coupling

1 Introduction

The goal of the present note is to detail the theoretical bases of models coupling Discrete Element Method (DEM) with Volume-Averaged Navier-Stokes (VANS) equations. It is general and describes 3D DEM-3D VANS coupling. It will be used as a basis for the description of YADE 1D vertical VANS fluid resolution in a subsequent note (12).

2 Fluid phase: Volume-Averaged Navier-Stokes equations

We start here from the equation of Jackson (10) and express the closures adopted consequently. Volume-averaging the local incompressible Navier-Stokes equations, Anderson & Jackson (1) obtained the following equations for a fluid phase in interactions with a particle phase:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_k}\left(\varepsilon \langle u_k\rangle^f\right) = 0 \tag{1}$$

$$\rho^f \varepsilon \left(\frac{\partial \langle u_i\rangle^f}{\partial t} + \langle u_k\rangle^f \frac{\partial}{\partial x_k}\langle u_i\rangle^f\right) = \frac{\partial S^f_{ik}}{\partial x_k} + \rho^f \varepsilon g_i - n\left\langle f^f_i\right\rangle^p, \tag{2}$$

Where ρ^f is the fluid density, $\varepsilon = 1-\phi$ is the fluid volume fraction, u_i is the i-th component of the velocity, $\langle\bullet\rangle^f$ and $\langle\bullet\rangle^p$ denote the volume averaging with respect to the fluid and particle phase respectively, S_{ij} is the total fluid stress tensor, g_i is the i-th component of the acceleration of gravity, and $n\left\langle f^f_i\right\rangle^p$ is the volume-averaged fluid-particle momentum transfer associated with the hydrodynamic forces (e.g. drag, buoyancy...). The expressions adopted here are the exact same as the one described in (10).

3 Closures

This equation requires closures for the total fluid stress tensor S^f_{ik} and the fluid-particle momentum transfer terms. The issues, the physical meaning and the choices made here for these closures are detailed in this section.

3.1 Total fluid stress tensor

Following Jackson (9; 10), the total fluid stress tensor reads:

$$S^f_{ij} = \underbrace{\varepsilon \left\langle \sigma_{ij} \right\rangle^f}_{1} + \underbrace{n \left\langle s^f_{ij} \right\rangle^p}_{2} + \underbrace{\frac{1}{2}\frac{\partial}{\partial x_k}\left(n \left\langle s^f_{ijk} \right\rangle^p \right)}_{3} \underbrace{-\rho^f \varepsilon \left\langle u'_i u'_j \right\rangle^f}_{4}, \tag{3}$$

where $\left\langle \sigma_{ij} \right\rangle^f$ is the stress tensor volume-averaged over the fluid phase, and $-\rho^f \varepsilon \left\langle u'_i u'_j \right\rangle^f$ is the cross velocity fluctuations volume-averaged over the fluid phase, classically called the Reynolds stress tensor.

3.1.1 Origin of the different terms...

The terms 1, 2 and 3 of equation (3) come from the Taylor expansion of the fluid-particle interaction term. Indeed, when volume-averaging the local Navier-Stokes equation on the fluid phase, it reads (9):

$$\rho^f \left[\frac{\partial \varepsilon \left\langle u_i \right\rangle^f}{\partial t} + \frac{\partial}{\partial x_k}(\varepsilon \left\langle u_i u_k \right\rangle^f) \right] = \frac{\partial (\varepsilon \left\langle \sigma_{ik} \right\rangle^f)}{\partial x_k} + \rho^f \varepsilon g_i - \sum_p \int_{S_p} \sigma_{ik}(\mathbf{y}) n_k(\mathbf{y}) g(|\mathbf{x}-\mathbf{y}|) dS_y. \tag{4}$$

The last term on the right hand side represents the momentum transfer from the fluid to the particles and is more complex than it appears. Classically the fluid force on a single particle is given from the integral of the traction force $t_i = \sigma_{ij} n_j$ on the surface of the particle, i.e.,

$$f^f_i = \int_{S_p} \sigma_{ij}(\mathbf{y})\, n_j\, dS_y. \tag{5}$$

In order to link the last term on the right hand side of equation (4) to the expression of the momentum transfer associated to the hydrodynamic forces, one needs to perform a Taylor expansion of the weighting function $g(|\mathbf{x}-\mathbf{y}|)$ around the center of the particle p. The latter expansion can be cut at second order thanks to the fact that the ratio between the radius of the particle, a, and the width L of volume-averaging function g, is small: $a/L << 1$ (scale separation). From this, we obtain (9):

$$\sum_p \int_{S_p} \sigma_{ik}(\mathbf{y}) n_k(\mathbf{y}) g(|\mathbf{x}-\mathbf{y}|) dS_y = n \left\langle f^f_i \right\rangle^p - n \frac{\partial}{\partial x_j} \left\langle s^f_{ij} \right\rangle^p + \frac{1}{2}\frac{\partial^2}{\partial x_j \partial x_l}\left(n \left\langle s^f_{ijl} \right\rangle^p \right) - ... + O(a^2/L^2), \tag{6}$$

where

$$n \left\langle f^f_i \right\rangle^p = \sum_p g(|\mathbf{x}-\mathbf{x}_p|) \int_{S_p} t_i(\mathbf{y})\, dS_y, \tag{7}$$

$$n \left\langle s^f_{ij} \right\rangle^p = a \sum_p g(|\mathbf{x}-\mathbf{x}_p|) \int_{S_p} t_i(\mathbf{y})\, n_j\, dS_y, \tag{8}$$

$$n \left\langle s^f_{ijl} \right\rangle^p = a^2 \sum_p g(|\mathbf{x}-\mathbf{x}_p|) \int_{S_p} t_i(\mathbf{y})\, n_j n_l\, dS_y. \tag{9}$$

so that the momentum balance equation of the fluid (eq. 4) becomes:

$$\rho^f \left[\frac{\partial \varepsilon \left\langle u_i \right\rangle^f}{\partial t} + \frac{\partial}{\partial x_k}(\varepsilon \left\langle u_i u_k \right\rangle^f) \right] = \frac{\partial (\varepsilon \left\langle \sigma_{ik} \right\rangle^f)}{\partial x_k} + \rho^f \varepsilon g_i - n \left\langle f^f_i \right\rangle^p + n \frac{\partial}{\partial x_j} \left\langle s^f_{ij} \right\rangle^p - \frac{1}{2}\frac{\partial^2}{\partial x_j \partial x_l}\left(n \left\langle s^f_{ijl} \right\rangle^p \right), \tag{10}$$

which corresponds exactly equation (2):

$$\rho^f \varepsilon \left(\frac{\partial \left\langle u_i \right\rangle^f}{\partial t} + \left\langle u_k \right\rangle^f \frac{\partial}{\partial x_k} \left\langle u_i \right\rangle^f \right) = \frac{\partial S^f_{ik}}{\partial x_k} + \rho^f \varepsilon g_i - n \left\langle f^f_i \right\rangle^p, \tag{11}$$

with the totall fluid stress tensor detailed above in equation (3):

$$S^f_{ij} = \underbrace{\varepsilon \left\langle \sigma_{ij} \right\rangle^f}_{1} + \underbrace{n \left\langle s^f_{ij} \right\rangle^p}_{2} + \underbrace{\frac{1}{2}\frac{\partial}{\partial x_k}\left(n \left\langle s^f_{ijk} \right\rangle^p \right)}_{3} \underbrace{-\rho^f \varepsilon \left\langle u'_i u'_j \right\rangle^f}_{4}, \tag{12}$$

Therefore, the terms 2 and 3 of the total fluid stress tensor corresponds to the perturbation of the fluid behavior due to the presence of the particles. This means that the particles affect the fluid not only from the momentum transfer associated with the fluid-particle force, but also from a rheological point of view, modifying the rheology by their presence in the fluid[1]. Jackson

[1] This can be seen as the stresslet in suspension

(9) has shown that these two terms reduce to the classical Einstein correction of the viscosity due to the presence of the particles in a fluid: $\mu_{eff} = \mu(1+5\phi/2)$. For more general regime however, there does not exist simple determination of these terms.

3.1.2 ...and closures

Volume-averaged fluid stress tensor The averaged fluid stress tensor can be directly linked to the volume-averaged pressure and shear rate. Indeed, the averaged fluid stress tensor expression reads from the definition of the average:

$$\left\langle \sigma_{ij} \right\rangle^f = \frac{1}{\varepsilon} \int_{V^f} \sigma_{ij}(\mathbf{y}) g(|\mathbf{x}-\mathbf{y}|) dV_y, \tag{13}$$

where the integral is over V^f, the whole volume occupied by the fluid. Considering a newtonian fluid, we can express the fluid stress tensor in this region:

$$\left\langle \sigma_{ij} \right\rangle^f = \frac{1}{\varepsilon} \int_{V^f} \left[-P^f \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] g(|\mathbf{x}-\mathbf{y}|) dy = - \langle P \rangle^f \delta_{ij} + \frac{\mu}{\varepsilon} \int_{V^f} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) g(|\mathbf{x}-\mathbf{y}|) dy, \tag{14}$$

where the expression of the averaged fluid pressure has been identified. In order to relate the expression of the second term on the right hand side of the equation to the averaged fluid velocity, one would like to exchange the integral and derivative sign. However, the volume occupied by the fluid is changing with space. Noting that the particles are considered rigid, the strain rate is null inside the volume occupied by the particles. We can then switch the integral from the volume occupied by the fluid V^f, to the total volume V:

$$\left\langle \sigma_{ij} \right\rangle^f = - \langle P \rangle^f \delta_{ij} + \frac{\mu}{\varepsilon} \int_{V} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) g(|\mathbf{x}-\mathbf{y}|) dy. \tag{15}$$

The latter does not vary so that we can exchange the integral sign and the derivative. This gives a rigourous closure for the volume-averaged fluid stress tensor:

$$\left\langle \sigma_{ij} \right\rangle^f = - \langle P \rangle^f \delta_{ij} + \frac{\mu}{\varepsilon} \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right), \tag{16}$$

where $\langle u_i \rangle$ is the averaged velocity defined as:

$$\langle u_i \rangle = \varepsilon \langle u_i \rangle^f + \phi \langle u_i \rangle^s . \tag{17}$$

This expression should not be confused with the more classical mixture velocity used in (10), which has a similar expression but is a density-weighted averaged.

Influence of the particles As already mentionned, Jackson (9) derived the expression of the terms 2 and 3 of equation (3) representing the influence of the particles on the fluid rheology, in the limit of dilute Stokesian particles ($St = 0$, $Re_p = 0$). In this case, it can be shown (see appendix A) that the terms $n \left\langle s_{ij} \right\rangle^p$ and $n \left\langle s_{ijk} \right\rangle^p$ reduce to the classical Einstein correction so that the total fluid stress tensor becomes:

$$S^f_{ij} = - \langle P \rangle^f \delta_{ij} + \mu \left(1 + \frac{5}{2}\phi \right) \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right) - \rho^f \varepsilon \left\langle u'_i u'_j \right\rangle^f . \tag{18}$$

Therefore, one could see the terms $n \left\langle s_{ij} \right\rangle^p$ and $n \left\langle s_{ijk} \right\rangle^p$ as a modification of the viscosity due to the presence of the particles. Even though the different assumptions made for the derivation of this expression are valid only for dilute stokesian suspension, we will consider that the effect of these terms are restricted to the modification of the viscosity of the volume-averaged fluid stress tensor. This is not strictly valid but this is consistent in the dilute stokesian limit and there exists today no real alternative. Therefore, the total stress tensor will be taken as:

$$S^f_{ij} = \sigma^{eff}_{ij} - \rho^f \varepsilon \left\langle u'_i u'_j \right\rangle^f , \tag{19}$$

with

$$\sigma^{eff}_{ij} = - \langle P \rangle^f \delta_{ij} + \mu f(\phi) \left(\frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle u_j \rangle}{\partial x_i} \right), \tag{20}$$

the effective fluid stress tensor accounting for the effect of the particles on the fluid rheology. Typical function for $f(\phi)$ are the first order Einstein's correction (5) or the second order Batchelor's correction (2). This function reprensents the influence of the particles on the fluid rheology, and not on the mixture rheology. Therefore, no contribution from contact-contact interactions should be included in the function f, which does not present any divergence at solid packing fraction. Typical expressions of suspension viscosity - which considers the viscosity of the mixture - such as Krieger-Dougherty viscosity are therefore not adapted.
In YADE, only Einstein's correction and clear fluid are implemented for now.

Reynolds stresses The last closure required to express the total fluid stress tensor, S_{ij}^f, is the volume-averaged fluid velocity fluctuations, $\rho^f \varepsilon \left\langle u_i' u_j' \right\rangle^f$. In a classical Reynolds Averaged Navier-Stokes (RANS) fluid resolution, the Reynolds stress tensor is the average over **time** of the fluid velocity fluctuations: $R_{ij}^f = -\rho^f \, \overline{u_i' u_j'}$. In the present case of two different phases in interaction, the average is made over space. Therefore, as done classically in the literature (3; 4; 8; 13; 15) we can adopt the classical closure for the Reynolds stress tensor such as mixing length, $k-\varepsilon$ or $k-\omega$. The volume-averaged fluid velocity fluctuation term will therefore be refered as the Reynolds stress tensor, $R_{ij}^f = -\rho^f \left\langle u_i' u_j' \right\rangle^f$, and the total fluid stress tensor becomes:

$$S_{ij}^f = \sigma_{ij}^{eff} + \varepsilon R_{ij}^f, \tag{21}$$

so that the fluid momentum balance equation (eq. 2) can be rewritten:

$$\rho^f \varepsilon \left(\frac{\partial \langle u_i \rangle^f}{\partial t} + \langle u_k \rangle^f \frac{\partial}{\partial x_k} \langle u_i \rangle^f \right) = \frac{\partial}{\partial x_k} \left(\sigma_{ik}^{eff} + \varepsilon R_{ij}^f \right) + \rho^f \varepsilon g_i - n \left\langle f_i^f \right\rangle^p \tag{22}$$

3.2 Momentum transfer associated to fluid-particle hydrodynamic force

The most obvious influence of the particles on the fluid momentum balance is the momentum transfered by the fluid phase to each individual particle through the classical hydrodynamic forces (e.g. drag, buoyancy, added mass...). The sum of the particles contributions is represented by $n \left\langle f_i^f \right\rangle^p$ in the fluid momentum balance (eq. 2). The latter corresponds to the particle number density n, multiplied by the hydrodynamic force volume-averaged over the particle phase $\left\langle f_i^f \right\rangle^p$. Therefore, one only needs to volume-average the fluid forces on the particles. However, the exact formulation of the force that a general fluid flow applies on a particle is not known. The exact derivation of this expression is possible only in the limit where the particle is isolated and particle Reynolds number tends to zero or to infinity (inviscid flows). This has been done in particular by Maxey and Riley (14) for the low particle Reynolds number case and by Schmeeckle et al. (17) in the inviscid flow limit.

We will not discuss here the expression of the force a particle undergoes in a fluid, but only point out the difficulty of it and the necessity to make choices and validate with experiments. The expression of the fluid force on a particle is therefore not known in intermediated particle Reynolds number regimes, which corresponds to the main applications of fluid-particle flows. This regime is therefore classically parametrized from empirical correlation in particular fluid flows, leading to a number of different formulations which can become increasingly complicated as for example for the drag force where some people considers power-laws of particle Reynolds number. In addition, the closures are usually determined for isolated particles, while real configurations such as fluidized bed, sediment transport, or immersed granular collapse for examples, include variation of particle concentration from isolated to densly packed. It is well known that the expression of the fluid-particle force adopted are drastically modified by the influence of the surrounding particles. For example, the drag force is known to be increased by the presence of other particles through the so-called hindrance effect, which has been related in the literature to the solid volume fraction (6; 16).

More generally, the complexity of fluid-particle flows and the multiplicity of approximated closures is such that DEM-VANS models are by essence approximative. As a consequence, adopting all the most complex closures for the fluid-particle do not guarantee to obtain physical results. Therefore and in order to understand the physical mechanisms at play, a simple and clear approach is to consider only the main fluid-particle forces at play and adopt simple empirical expressions for these forces. Doing so, it is then necessary to compare the model results to experiment in order to ensure that the closures and the fluid-particle forces adopted reproduce accurately the physics of the problem considered.
For example, considering the case of bedload transport, in the model developed by the author (13), the main forces at play are the drag, the buoyancy and the lift forces (11). However, the actual expression of the lift force is controversial and has been

shown to be not well adapted to the case of bedload transport (17). Therefore, we have chosen to restricts ourselves to drag and buoyancy forces. The comparison of the model with the particle-scale experiments of Frey (7) has shown that the forces and closures adopted allows to reproduce very finely the granular behavior (13), validating the choices made. This, of course, restricts the range of possible applications of the model, but corresponds to my opinion to the appropriate approach considering the actual limited state of knowledge on fluid-particle forces expression and more generally considering all the closures of VANS model.

A Influence of the particles on the rheology: example of a stokesian dilute suspension

In the case of Stokesian dilute suspension, Jackson (9) has derived the expression of the influence of the particles on the fluid rheology (terms 2 and 3 of eq. 3). The goal of the present appendix is to reproduce this derivation, in order to understand the physical meaning of these terms.
Starting from the momentum balance of a fluid in interaction with a particle phase (equation 2):

$$\rho^f \varepsilon \left(\frac{\partial \langle u_i \rangle^f}{\partial t} + \langle u_k \rangle^f \frac{\partial}{\partial x_k} \langle u_i \rangle^f \right) = \frac{\partial S^f_{ik}}{\partial x_k} + \rho^f \varepsilon g_i - n \left\langle f^f_i \right\rangle^p$$
$$= \frac{\partial}{\partial x_k} \left\{ \varepsilon \langle \sigma_{ij} \rangle^f + n \left\langle s^f_{ij} \right\rangle^p + \frac{1}{2} \frac{\partial}{\partial x_k} \left(n \left\langle s^f_{ijk} \right\rangle^p \right) - \rho^f \varepsilon \langle u'_i u'_j \rangle^f \right\} + \rho^f \varepsilon g_i - n \left\langle f^f_i \right\rangle^p \quad (23)$$

and considering the expression of the traction force at the surface of the particle from (9), one can calculate explicitely $n \left\langle s^f_{ij} \right\rangle^p$ and $n \left\langle s^f_{ijk} \right\rangle^p$ (9):

$$n \left\langle s^f_{ij} \right\rangle^p = \phi \left\{ - \langle P \rangle^f \delta_{ij} + 3\mu \left[\frac{1}{2} \left(\frac{\partial \langle u_i \rangle^f}{\partial x_j} - \frac{\partial \langle u_j \rangle^f}{\partial x_i} \right) - \varepsilon_{ilj} \langle \omega_l \rangle^p \right] \right.$$
$$\left. + \frac{5\mu}{2} \left(\frac{\partial \langle u_i \rangle^f}{\partial x_j} + \frac{\partial \langle u_j \rangle^f}{\partial x_i} \right) \right\} + O(\phi^2) \; [+O(a^2/L^2)] \quad (24)$$

$$n \left\langle s^f_{ijk} \right\rangle^p = \phi \left\{ \frac{3\mu}{2} \left(\langle u_i \rangle^f - \langle u_i \rangle^p \right) \delta_{jk} - \frac{a^2 \rho^f}{5} \left(g_i \delta_{jk} + g_j \delta_{ki} + g_k \delta_{ij} \right) + \mu a^2 C^{abc}_{ijk} \frac{\partial^2 \langle u_a \rangle^f}{\partial x_b \partial x_c}, \right\} + O(\phi^2)$$
$$= \phi \frac{3\mu}{2} \left(\langle u_i \rangle^f - \langle u_i \rangle^p \right) \delta_{jk} + O(\phi^2) + O(a^2/L^2) \quad (25)$$

where C^{abc}_{ijk} *"represents a sum of various second derivatives of the fluid-phase average velocity"*, which are of order a^2/L^2 with respect to the other second derivative at play in the momentum balances of the fluid and of the mixture. Neglecting the Reynolds stress tensor terms (as the suspension is dilute and stokesian, see the discussion p. 8 of (9)) and the terms of order greater than a^2/L^2, equation (23) becomes:

$$\rho^f \varepsilon \left(\frac{\partial \langle u_i \rangle^f}{\partial t} + \langle u_k \rangle^f \frac{\partial}{\partial x_k} \langle u_i \rangle^f \right) = \frac{\partial}{\partial x_k} \left\{ \varepsilon \langle \sigma_{ik} \rangle^f - \phi \langle P \rangle^f \delta_{ij} \right.$$
$$+ 3\mu \left[\frac{1}{2} \left(\frac{\partial \langle u_i \rangle^f}{\partial x_j} - \frac{\partial \langle u_j \rangle^f}{\partial x_i} \right) - \varepsilon_{ilj} \langle \omega_l \rangle^p \right] + \frac{5\mu}{2} \left(\frac{\partial \langle u_i \rangle^f}{\partial x_j} + \frac{\partial \langle u_j \rangle^f}{\partial x_i} \right)$$
$$\left. - \frac{1}{2} \frac{\partial}{\partial x_k} \left(\phi \frac{3\mu}{2} \left(\langle u_i \rangle^f - \langle u_i \rangle^p \right) \delta_{jk} \right) \right\} + \rho^f \varepsilon g_i - n \left\langle f^f_i \right\rangle^p . \quad (26)$$

The volume-averaged fluid stress tensor reads (see section 3.1.2):

$$\varepsilon \left\langle \sigma_{ij} \right\rangle^f = -\varepsilon \left\langle P \right\rangle^f \delta_{ij} + \mu \left(\frac{\partial \left\langle u_i \right\rangle}{\partial x_j} + \frac{\partial \left\langle u_j \right\rangle}{\partial x_i} \right). \tag{27}$$

In addition, when considering dilute stokesian suspension, the fluid-particle velocity slip is negligible and the antisymetric part of the stress tensor is negligible. Therefore, the equation can be rewritten:

$$\begin{aligned}\rho^f \varepsilon \left(\frac{\partial \left\langle u_i \right\rangle^f}{\partial t} + \left\langle u_k \right\rangle^f \frac{\partial}{\partial x_k} \left\langle u_i \right\rangle^f \right) &= \frac{\partial}{\partial x_k} \left\{ \varepsilon \left\langle \sigma_{ik} \right\rangle^f - \phi \left\langle P \right\rangle^f \delta_{ij} + \frac{5\mu}{2} \left(\frac{\partial \left\langle u_i \right\rangle^f}{\partial x_j} + \frac{\partial \left\langle u_j \right\rangle^f}{\partial x_i} \right) \right\} + \rho^f \varepsilon g_i - n \left\langle f_i^f \right\rangle^p \\ &= \frac{\partial}{\partial x_k} \left\{ -\varepsilon \left\langle P \right\rangle^f \delta_{ij} + \mu \left(\frac{\partial \left\langle u_i \right\rangle}{\partial x_j} + \frac{\partial \left\langle u_j \right\rangle}{\partial x_i} \right) - \phi \left\langle P \right\rangle^f \delta_{ij} + \frac{5\mu}{2} \left(\frac{\partial \left\langle u_i \right\rangle^f}{\partial x_j} + \frac{\partial \left\langle u_j \right\rangle^f}{\partial x_i} \right) \right\} + \rho^f \varepsilon g_i - n \left\langle f_i^f \right\rangle^p .\end{aligned} \tag{28}$$

In the case of stokesian suspension, the slip velocity is negligible so that:

$$\left\langle u_i \right\rangle = \varepsilon \left\langle u_i \right\rangle^f + \phi \left\langle u_i \right\rangle^p = \left\langle u_i \right\rangle^f + \phi \left(\left\langle u_i \right\rangle^p - \left\langle u_i \right\rangle^f \right) \simeq \left\langle u_i \right\rangle^f . \tag{29}$$

As a consequence and adding the two pressure terms, we obtain the following equation for the fluid momentum balance:

$$\rho^f \varepsilon \left(\frac{\partial \left\langle u_i \right\rangle^f}{\partial t} + \left\langle u_k \right\rangle^f \frac{\partial}{\partial x_k} \left\langle u_i \right\rangle^f \right) = \frac{\partial}{\partial x_k} \left\{ - \left\langle P \right\rangle^f \delta_{ij} + \mu \left(1 + \frac{5}{2}\phi \right) \left(\frac{\partial \left\langle u_i \right\rangle}{\partial x_j} + \frac{\partial \left\langle u_j \right\rangle}{\partial x_i} \right) \right\} + \rho^f \varepsilon g_i - n \left\langle f_i^f \right\rangle^p . \tag{30}$$

This equation is established for a dilute case with Stokesian particles. It shows that in this case, the presence of the particles affects the fluid through both the momentum transfer associated to hydrodynamic forces and the influence of the particles on the effective fluid rheology. The presence of the particles modifies the pressure terms - which is not anymore multiplied by the fluid solid volume fraction - and the viscosity - which accounts for Einstein's correction (5) . In addition, the strain rate tensor is not the one of the fluid but of the combination of fluid and particles.

In the following, the effective fluid stress tensor accounting for both the fluid contribution and the effect of the particle on the fluid rheology:

$$\sigma_{ij}^{eff} = \varepsilon \left\langle \sigma_{ij} \right\rangle^f + n \left\langle s_{ij}^f \right\rangle^p + \frac{1}{2} \frac{\partial}{\partial x_k} \left(n \left\langle s_{ijk}^f \right\rangle^p \right), \tag{31}$$

will be taken as:

$$\sigma_{ij}^{eff} = - \left\langle P \right\rangle^f \delta_{ij} + \mu f(\phi) \left(\frac{\partial \left\langle u_i \right\rangle}{\partial x_j} + \frac{\partial \left\langle u_j \right\rangle}{\partial x_i} \right), \tag{32}$$

in order to recover the right limit when considering stokesian dilute suspensions.

Acknowledgements

I want to thank Laurent Lacaze for interesting discussions on the subject, that helped me to clarify some aspects.

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